A Frequency Domain Approach to Linear Optimal Control

R. F. WHITBECK*
Cornell Aeronautical Laboratory Inc., Buffalo, N.Y.

To find the control that minimizes a quadratic performance index requires, in the frequency domain, the solution of a matrix Wiener-Hopf equation. A simple algebraic method is given for solving this equation in the linear multicontroller case. The solution yields not only the optimal control but the feedback gains as well. This obviates the need to solve a steady-state Riccati equation. The ideas involved are illustrated with a simplified set of equations of motion representing a small jet aircraft in a power approach. Thrust and elevator are both used as active controllers. A method that specifies a set of desirable closed-loop poles and yet retains some freedom for positioning the zeros of the system is discussed.

Introduction

MOST approaches to the linear optimal control problem depend on time domain techniques for the establishment of their theoretical foundation. In contrast, this paper discusses a frequency domain method that leads to the formulation of a matrix Wiener-Hopf equation. The approach has practical significance when applied to a regulator problem for at least two good reasons: 1) The matrix Wiener-Hopf equation is easily solved using only algebraic methods. 2) The solution of the matrix Wiener-Hopf equation yields not only the optimal control but the feedback gains as well. The second point deserves further elaboration. The optimal control vector is obtained as a function of s and the initial condition vector $x(0)$. That is,

$$U_0[s,x(0)] = C(s)x(0)$$

where C is a matrix. Applying the initial value theorem to this equation leaves, as the only surviving entries in the matrix, the feedback gains required by the control law: $u_0(t) = -Kx(t)$ since $u_0(0) = -Kx(0)$. Thus the feedback gains are obtained by inspection once $U_0(s)$ is known, and the sometimes painful chore of waiting for a digital computer to converge to the steady-state solution of a Riccati equation can be avoided.

A detailed derivation of the matrix Wiener-Hopf equation will not be given since it is available in the literature (e.g., Ref. 1). However, a "shorthand" method for deriving these equations is given since the method is simple, useful, and hitherto unavailable in the literature.

An illustrative example is restricted to the two-controller case and has a minimum complexity that is consistent with demonstrating the procedures involved (e.g., the numbers used are simple enough so that the interested reader may easily verify the manipulations using only a pad and pencil). In some applications, the engineer uses linear optimal control techniques only after he has investigated the location of the closed-loop poles (as a function of the Q and R matrices) with some such analysis aid as the root square locus.[1] Thus, what is really occurring is that the decisions as to which q's and r's are to be used depend on where the designer feels the closed-loop poles should be located. In such applications, the question arises as to whether or not it is necessary at all to go through the intermediate step involving the q's and r's when what is really desired is a specified set of closed-loop poles. The answer, of course, is that the tools of linear optimal control can be used to achieve a specified set of closed-loop poles without finding the performance index parameters which force the situation. The last example of the paper demonstrates a method for doing this. The example utilizes a small jet aircraft, in which both throttle (thrust) and elevator are used as active controllers, as the open-loop plant. The reader is cautioned that this last example is carried through with a degree of numerical precision that many will feel is unjustified. This numerical precision is strictly for the benefit of those readers who are interested in carefully checking the assertions that are being made.

Received November 8, 1967; revision received March 20, 1968.

* Principal Engineer, Flight Research Department.

Wiener-Hopf Equation

For this problem the differential equations of aircraft motion are assumed to have constant coefficients and can be written as a matrix set of first-order equations of the form

$$\dot{x} = Fx + Gu + a(t) \qquad y = Hx \tag{1}$$

where

x = the vector whose components are the variables of the differential equations of motion
u = the control vector
$a(t)$ = disturbance input vector
y = the output; a transformed set on the state whose motions are to be minimized
F = an $n \times n$ matrix of constants describing the coupling among state variables in the equations of motion, and
G = an $n \times p$ matrix of constants describing the effect of control inputs on the dynamic motions of the system

A control motion u_0 is to be found that minimizes the quadratic performance index:

$$J = \int_0^\infty (y'Qy + u'Ru)dt \tag{2}$$

Q = $r \times r$ symmetric matrix whose elements weight the contributions of each output in the integral
R = a $p \times p$ symmetric matrix whose elements weigh the contribution of each control motion in the integral

Applying Parseval's theorem to Eq. (2) gives

$$J = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \{Y_*QY + U_*RU\}ds \tag{3}$$

where

$$\begin{aligned} Y &= Y(s) = \mathfrak{L}[y(t)] \\ Y_* &= Y'(-s) = \text{the transpose of the vector } Y(-s) \end{aligned} \tag{4}$$

Taking the two-sided Laplace transform of Eq. (1) yields†

$$Y = H(Is - F)^{-1}GU + H(Is - F)^{-1}[x(0) + A(s)] \quad (5)$$

For the sake of brevity, write

$$Y(s) = W(s)U(s) + B(s) \quad (6)$$

so that

$$Y_* = [Y(-s)]' = U_*W_* + B_* \quad (7)$$

The expression for Y_*QY becomes

$$Y_*QY = [U_*W_* + B_*]Q[WU + B] \quad (8)$$

Substituting Eq. (8) into Eq. (3) gives

$$2V = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \{[U_*W_* + B_*]Q[WU + B] + U_*RU\}\, ds \quad (9)$$

We seek the optimum control vector which minimizes the performance index of Eq. (9). A lengthy and formal procedure for doing this is given in Ref. 1.

Rather than use this longer procedure, we will resort to a shorthand technique that is quite general and very useful. In S. S. L. Chang's book (Ref. 2, p. 71), the following variational method for deriving a Wiener-Hopf equation is given. Let

$$J = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \phi(\bar{m},m)ds \quad (10)$$

where

$$m = m(s) \qquad \bar{m} = m(-s) \quad (11)$$

That is, the integrand of the performance index satisfies a "conjugate" condition with respect to a scalar m. The Wiener-Hopf equation that defines the value of m which minimizes Eq. (10) is

$$\partial\phi/\partial\bar{m} = z(s) \quad (12)$$

where $\mathcal{L}^{-1}[z(s)]$ is required to be a function of time which is zero for $t \geq 0$. The use of negative and positive time functions is what necessitates the use of the bilateral Laplace transform. This point is discussed in Ref. 3.

Although Chang did not use a vector notation, his work is extensive enough to make the rephrasing of the method in vector form a straightforward task. Let M be a vector and

$$J = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \Phi(M_*,M)ds \quad (13)$$

where

$$M = M(s) \qquad M_* = M'(-s) \quad (14)$$

The Wiener-Hopf equation that defines the value of the vector M which minimizes Eq. (13) is the vector equation

$$\partial\Phi/\partial M_* = Z(s) \quad (15)$$

where Φ is a scalar and $Z(s)$ is a vector which is required to give rise to functions of time which exist only for negative time.

Return now to Eq. (9) and write

$$\Phi = (U_*W_* + B_*)Q(WU + B) + U_*RU \quad (16)$$

Since the minimization is with respect to the control vector U we obtain

$$\partial\phi/\partial U_* = W_*Q[WU + B] + RU = Z(s)$$

or

$$[R + W_*QW]U_0 + W_*QB = Z(s) \quad (17)$$

Equation (17) is the correct vector Wiener-Hopf equation.[1] It is frequently called a matrix Wiener-Hopf equation because the matrix $[R + W_*QW]$ and its spectral factorization are basic to all the known techniques used for its solution. It is easy to show (and probably obvious to the reader anyway) that the left half-plane poles of

$$\det[R + W_*QW]$$

represent the closed-loop roots of the system.‡ Let

$$\det[R + W_*QW] = \Delta(s)\Delta(-s) = \Delta\bar{\Delta} \quad (18)$$

Equation (18) states that $\det[R + W_*QW]$ is a polynomial in s^2 and therefore symmetric in the s plane, about the imaginary as well as the real axis. In the next section, an algebraic method for solving Eq. (17) will be given. Before proceeding with this, however, we will take the opportunity to summarize the important equations for the multicontrol case: Given

$$\dot{x}(t) = Fx(t) + Gu(t) + a(t) \quad (19)$$

and

$$y(t) = Hx(t) \quad (20)$$

$$J = \int_0^\infty (y'Qy + u'Ru)dt \quad (21)$$

Let

$$H[Is - F]^{-1}G = W \qquad \text{a matrix} \quad (22)$$

$$G'[-Is - F']^{-1}H' = W_* \qquad \text{a matrix} \quad (23)$$

$$H[Is - F]^{-1}[x(0) + A] = B \qquad \text{a column vector} \quad (24)$$

$$[x'(0) + A'(-s)][-Is - F']^{-1}H' = B_* \qquad \text{a row vector} \quad (25)$$

so that

$$Y(s) = W(s)U(s) + B(s) \quad (26)$$

The vector Wiener-Hopf equation is then

$$[R + W_*QW]U_0 + W_*QB = Z \quad (27)$$

An Algebraic Method for Solving the Matrix Wiener-Hopf Equation

Because of the simplicity of the technique, the essential details of the algebraic approach can be stated in a very succinct manner. Each component of $Z(s)$ can be written as the ratio of two polynomials in s since the control vector itself (which is of dimension p) must have the form

$$\begin{bmatrix} U_{01} \\ U_{02} \\ \cdot \\ \cdot \\ \cdot \\ U_{0p} \end{bmatrix} = \begin{bmatrix} a_1/\Delta\Gamma \\ a_2/\Delta\Gamma \\ \cdot \\ \cdot \\ \cdot \\ a_p/\Delta\Gamma \end{bmatrix} \quad (28)$$

In Eq. (28), Γ represents the poles of the input vector $A(s)$, whereas Δ represents the poles of the closed-loop system. For a given set of Q and R, Δ is obtained by factoring the $\det[R + W_*QW]$ into the product of a left and right half-plane component. Also, $a_1, a_2 \ldots, a_p$ represent polynomials (with unknown coefficients) of one order less than the order

† The reason for the bilateral transform is discussed in great length in Ref. 3. The theoretical simplifications that occur from its use will be mentioned at appropriate points in the text.

‡ The optimal control itself must also contain the poles of $A(s)$, the disturbance input vector. However, it is customary to consider these as poles of the excitation rather than representing any basic property of the system itself.

of $\Delta\Gamma$. This order must be at least one less since $U_0(s)$ appears in the performance index. To insure the existence of this index, $U_0(s)$ must be a proper rational function.[1]

Using Eq. (28), Eq. (27) can be rewritten as

$$\begin{bmatrix} \xi_1(s)/\Gamma\Delta D\bar{D} \\ \xi_2(s)/\Gamma\Delta D\bar{D} \\ \cdot \\ \cdot \\ \xi_p(s)/\Gamma\Delta D\bar{D} \end{bmatrix} = \begin{bmatrix} Z_1(s) \\ Z_2(s) \\ \cdot \\ \cdot \\ Z_p(s) \end{bmatrix} \tag{29}$$

In Eq. (29), D is a polynomial that represents the poles of the open-loop plant. The poles of D and Γ may, of course, be anywhere in the complex plane but the poles of Δ must be in the left ha lf-plane.[3]

Since $\mathcal{L}^{-1}[Z(s)] = 0$ for $t \geq 0$, it is apparent that the numerators $\xi_1(s), \xi_2(s), \ldots \xi_p(s)$ must contain factors that identically cancel all those poles that can produce positive time functions. That is, all the polynomials $\xi_1, \xi_2, \ldots \xi_p$, must contain the factor $\Gamma\Delta D$. This leads to the requirement that

$$\left.\begin{aligned} \xi_1(s) &= 0 \\ \xi_2(s) &= 0 \\ \cdot &= \\ \cdot &= 0 \\ \cdot & \\ \xi_p(s) &= 0 \end{aligned}\right\} \quad \text{when } \Delta D\Gamma = 0 \tag{30}$$

In short, $U_0(s)$ must be found so that $Z(s)$ is finite for the values of s satisfying $\Delta D\Gamma = 0$:

$$[R + W_*QW]U_0 + W_*QB \neq \infty \quad \text{when} \quad \Delta D\Gamma = 0 \tag{31}$$

This procedure produces more equations than unknowns, but linear dependence will always reduce this set to the proper number of equations.[1] Note that there is a total of $p(n + m)$ unknowns in the numerators of the p components of the optimal-control vector, if one specifies the order of Δ as n and the order of Γ as m.

Equation (31) can be simplified considerably by studying separately the equations that arise when $D(s) = 0$, $\Delta(s) = 0$, and $\Gamma(s) = 0$. For those values of s where $D = 0$, the RU_0 term in Eq. (31) is of no interest, since $U_0 \neq \infty$ when $D = 0$. Therefore Eq. (31) reduces to

$$W_*Q[WU_0 + B] \neq \infty \quad \text{when} \quad D = 0 \tag{32}$$

but $W_*Q = \infty$ only when $\bar{D} = 0$; therefore, it also is of no concern. Thus, satisfying (31) reduces to

$$WU_0 + B \neq \infty \quad \text{when} \quad D = 0 \tag{33}$$

which is independent of Q and R. This means, in the case of an nth-order open-loop system with p controllers, that n equations in terms of the $p(m + n)$ unknown coefficients of the optimal control vector can be found using one relatively simple equation.

Next, consider Eq. (31) when $\Delta(s) = 0$. W_*QB does not depend on Δ since it equals infinity only when $D\bar{D} = 0$. The condition therefore becomes

$$[R + W_*QW]U_0 \neq \infty \quad \text{when} \quad \Delta = 0 \tag{34}$$

The equations arrived at using this condition *do* depend explicitly on R and Q.

Finally consider $\Gamma(s) = 0$. Since $U_0(s) \to \infty$ when $\Gamma(s) \to \infty$ as well as W_*QB, it is apparent that we must consider Eq. (31) in its totality, and no simplification results. To summarize, one may generate the required equations in the $p(n + m)$ unknowns by imposing the following requirements on the equations:

$$WU_0 + B \neq \infty \quad \text{when} \quad D = 0 \tag{35}$$

$$[R + W_*QW]U_0 \neq \infty \quad \text{when} \quad \Delta = 0 \tag{36}$$

$$[R + W_*QW]U_0 + W_*QB \neq \infty \quad \text{when} \quad \Delta D\Gamma = 0 \tag{37}$$

It is apparent that a considerable simplification occurs in the regulator case since $A(s)$ [refer to Eq. (24)] is identically zero and Eq. (37) is of no interest. The next section considers this case in detail when there are two controllers.

Regulator Problem

In this section, an example will be used to demonstrate the use of Eqs. (35) and (36) in solving for the optimal control in the regulator case. The reasons for considering the two-controller case are numerous. First, it would be unrewarding to consider only one controller since the matrix Wiener-Hopf equation would reduce to a scalar and this can be solved completely using only Eq. (35).

The two-controller case is the one of lowest order (in terms of the number of controllers) which exhibits all the complexity one encounters in the multicontroller problem and hence all the techniques required may be demonstrated.

As mentioned earlier, the use of Eqs. (35) and (36) will produce more equations than unknowns, but it is usually obvious which are linearly dependent. For example, $\det[R + W_*QW] = 0$ when $\Delta\bar{\Delta} = 0$ requires, in Eq. (36), that the equations produced by using the $Z_1(s)$ component of Z must be proportional to those found using the $Z_2(s)$ component in the two-controller case. Similar comments hold true for Eq. (35) since both W and B contain $(Is - F)^{-1}$, whose determinant is zero when $D = 0$.

Similarly, in the three-controller case, one may consider any two out of the three components of $Z(s)$ to obtain the necessary equations. An illustrative example is in order (the numbers are simple enough for the interested reader to follow easily the manipulations):

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = Fx + GU \tag{38}$$

Further, let

$$H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad Q = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} \qquad R = \begin{bmatrix} \frac{1}{7} & 0 \\ 0 & \frac{1}{4} \end{bmatrix} \tag{39}$$

Taking the Laplace transform of Eq. (38) and solving for $Y(s)$ gives

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \frac{\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}}{(s+1)(s+2)} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} + \frac{\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}}{(s+1)(s+2)} \times \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = WU + B \tag{40}$$

Also

$$[R + W_*QW] = \frac{\begin{bmatrix} \frac{1}{7}(s^4 - 5s^2 + 4) + 20 & -10s \\ 10s & \frac{1}{4}(s^4 - 5s^2 + 4) - 5s^2 \end{bmatrix}}{(s^4 - 5s^2 + 4)} \tag{41}$$

Since $WU_0 + B \neq \infty$ when $D = 0$, we may require that $Y_1(s) \neq \infty$ when $D = 0$:

$$(s + 3)U_1 + U_2 + (s + 3)x_1(0) + x_2(0) = 0 \quad \text{when } s = -1, -2 \tag{42}$$

Let

$$U_{01} = (a_0s + a_1)/\Delta \qquad U_{02} = (b_0s + b_1)/\Delta \tag{43}$$

Table 1 Aerodynamic derivatives

V_T, fps	α_T, rad	$D\alpha$	$D\delta_T$	D_V	L_V	L_θ
257	0.209	−0.0679	−0.0329	0.0296	−530.2.0260−	
$L\alpha$	$L\delta_e$	$M\alpha$	$M\dot{\alpha}$	M_V	M_q	$M\delta_e$
−0.667	−0.0326	−1.75	−0.215	0.000366	−0.536	−2.66

where a_0, a_1, b_0, and b_1 are undetermined coefficients. The closed-loop characteristic equation of the system Δ is found by evaluating the determinant of $[R + W_*QW]$ and finding the left half-plane roots:

$$\Delta = s^2 + 7s + 12 = (s+3)(s+4) \tag{44}$$

Substituting Eqs. (43) and (44) gives

$$(s+3)(a_0 s + a_1) + (b_0 s + b_1) + \Delta[(s+3)x_1(0) + x_2(0)] = 0 \quad \text{when } s = -1, -2 \tag{45}$$

In Eq. (45), let $s = -1, -2$, and place the result in matrix form. When $D = 0$

$$\begin{bmatrix} 2 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} 12 & 6 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \tag{46}$$

The requirement $[R + W_*QW]U_o \neq \infty$ when $\Delta = 0$ now gives, using Eq. (41),

$$[\tfrac{1}{7}(s^4 - 5s^2 + 4) + 20](a_0 s + a_1) - 10s(b_0 s + b_1) = 0 \quad \text{when } \Delta = 0 \tag{47}$$

In Eq. (47), let $s = -3, -4$, and place the result in matrix form. When $\Delta = 0$

$$\begin{bmatrix} -3 & 1 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} + \begin{bmatrix} -\frac{7}{2} & \frac{7}{6} \\ -\frac{7}{2} & \frac{7}{8} \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = 0 \tag{48}$$

Note that Eq. (48) is autonomous, whereas Eq. (46) depends explicitly on the initial condition vector.

Solving Eqs. (46) and (48), one may show rather easily that

$$\begin{bmatrix} a_0 \\ a_1 \\ b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} -\frac{7}{4} & \frac{7}{8} \\ -\frac{21}{2} & -\frac{7}{4} \\ \frac{1}{2} & -\frac{9}{4} \\ 6 & -3 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \tag{49}$$

Now consider the optimal control law, in the regulator case[1] which is

$$U_0 = -Kx(s) \tag{50}$$

Applying the initial value theorem to both sides of Eq. (50) gives

$$\lim_{s\to\infty} SU_{0[s,x(0)]} = \lim_{s\to\infty} \begin{bmatrix} (a_0 s^2 + a_1 s)/\Delta \\ (b_0 s^2 + b_1 s)/\Delta \end{bmatrix} = \begin{bmatrix} a_0 \\ b_0 \end{bmatrix} = -Kx(0) \tag{51}$$

Equation (51) shows that once the optimal control has been defined as a function of s and the initial condition vector $[x(0)]$, one may find the feedback gains by inspecting the optimal control solution. For this example,

$$K = \begin{bmatrix} \frac{7}{4} & -\frac{7}{8} \\ -\frac{1}{2} & \frac{9}{4} \end{bmatrix} \tag{52}$$

$U_0[s, x(0)]$ may also be written out in the explicit form given below:

$$\begin{bmatrix} U_{01} \\ U_{02} \end{bmatrix} = \frac{\begin{bmatrix} -(\frac{7}{4}s + \frac{21}{2}) & \frac{7}{8}s - \frac{7}{4} \\ (\frac{1}{2}s + 6) & -(\frac{9}{4}s + 3) \end{bmatrix}}{(s+3)(s+4)} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \tag{53}$$

This example considered only real open- and closed-loop roots. If the roots are complex conjugates for example, $s = -\alpha \pm i\beta$, it is obvious that the procedures described previously go through smoothly. It is only necessary to equate the appropriate real component equal to zero and the appropriate imaginary component equal to zero. For multiple roots, the numerators in Eq. (29) must contain factors like $(s + \alpha)^2$. Therefore, the numerators as well as their derivatives must equal zero when $s = -\alpha$. Both of these cases will arise in the next example.

Aircraft Two-Controller Problem

The previous example was designed to demonstrate technique and hence was restricted to a second-order system in order that the numbers would be tractable. Here, something that more closely resembles a real problem will be studied. Specifically, a small jet transport in a powered approach flight condition will be considered. The linearized equations of motion, in a body axis reference system for straight and level flight (as well as smooth air), are assumed to be given by the following (V, α, θ represent perturbations from a trim condition):

Drag equation:

$$\dot{V} + D_V V + V_T D_\alpha \alpha + g\theta = -V_T D_{\delta_T} \delta_T \tag{54}$$

Lift equation:

$$(\alpha_T/V_T)\dot{V} - (1/V_T)L_V V + \dot{\alpha} - L_\alpha \alpha - \dot{\theta} - L_\theta \theta = L_{\delta_e}\delta_e \tag{55}$$

Pitching moment equation:

$$-M_V V - M_{\dot{\alpha}}\dot{\alpha} - M_\alpha \alpha + \ddot{\theta} - M_q \dot{\theta} = M_{\delta_e}\delta_e \tag{56}$$

It is perhaps a little tricky to place the aforesaid set in first-order form. To do this, use as the state vector

$$x(s) = \begin{bmatrix} \dot{\theta} \\ \theta \\ V \\ \alpha \end{bmatrix} \tag{57}$$

First, solve Eq. (54) for $\dot{V}$ and use this as one equation. Next, solve Eq. (55) for $\dot{\alpha}$ (eliminate $\dot{V}$ terms) and use this as a second equation. Finally, solve Eq. (56) for $\ddot{\theta}$, eliminating the $\dot{\alpha}$ term (which, in turn requires the elimination of $\dot{V}$ terms). These three equations, together with the identity, $\dot{\theta} = \dot{\theta}$, give the following first-order form:

$$\begin{bmatrix} \ddot{\theta} \\ \dot{\theta} \\ \dot{V} \\ \dot{\alpha} \end{bmatrix} = \begin{bmatrix} (M_q + M_{\dot{\alpha}}) & (M_{\dot{\alpha}}/V_T)(g\alpha_T + V_T L_\theta) & (M_{\dot{\alpha}}/V_T)(\alpha_T D_V + L_V) + M_V & M_\alpha + M_{\dot{\alpha}}(\alpha_T D_\alpha/V_T) + (M_{\dot{\alpha}} L_\alpha/V_T) \\ 1 & 0 & 0 & 0 \\ 0 & -g & -D_V & -V_T D_\alpha \\ 1 & (1/V_T)(g\alpha_T + V_T L_\theta) & (1/V_T)(\alpha_T D_V L_V) & (\alpha_T D_\alpha L_\alpha) \end{bmatrix} \times \begin{bmatrix} \dot{\theta} \\ \theta \\ V \\ \alpha \end{bmatrix} + \begin{bmatrix} M_{\dot{\alpha}}\alpha_T D_{\delta T} & M_{\dot{\alpha}} L_{\delta_e} + M_{\delta_e} \\ 0 & 0 \\ -V_T D_{\delta T} & 0 \\ \alpha_T D_{\delta T} & L_{\delta_e} \end{bmatrix} \begin{bmatrix} \delta_T \\ \delta_e \end{bmatrix} \tag{58}$$

A typical set of the aerodynamic derivatives are given in the Table 1.[1]

Using these derivatives, the first-order equation, Eq. (58), becomes

$$\begin{bmatrix} \ddot{\theta} \\ \dot{\theta} \\ \dot{V} \\ \dot{\alpha} \end{bmatrix} = \begin{bmatrix} -0.751 & 0.000003 & 0.000572 & -1.6 \\ 1.000 & 0 & 0 & 0 \\ 0 & -32.2 & -0.0296 & 17.45 \\ 1.000 & -0.000014 & -0.0009599 & -0.681 \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \theta \\ V \\ \alpha \end{bmatrix} + \begin{bmatrix} 0.0015 & -2.65 \\ 0 & 0 \\ 8.46 & 0 \\ -0.0069 & -0.0326 \end{bmatrix} \begin{bmatrix} \delta_T \\ \delta_e \end{bmatrix} \tag{59}$$

There are several points to be illustrated which would probably be obscured by working with the system defined by Eq. (59). That is, the fine points might be lost in a host of computational difficulties associated with the wide range of coefficient values contained in the equation. For this reason, we will simplify by throwing out all the small terms as well as changing others to numerically convenient values. After the procedures are well in hand, the interested reader may, of course, repeat them using Eq. (59). Consider now the fictitious airplane described by

$$\begin{bmatrix} \ddot{\theta} \\ \dot{\theta} \\ \dot{V} \\ \dot{\alpha} \end{bmatrix} = \begin{bmatrix} -1.2 & 0 & 0 & -1.04 \\ 1 & 0 & 0 & 0 \\ 0 & -32 & 0 & 20 \\ 1 & 0 & 0 & -0.8 \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \theta \\ V \\ \alpha \end{bmatrix} + \begin{bmatrix} 0 & -3 \\ 0 & 0 \\ 10 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_T \\ \delta_e \end{bmatrix} \tag{60}$$

Using Eq. (60), the matrices necessary for the solution are

$$[Is - F] = \begin{bmatrix} s + 1.2 & 0 & 0 & 1.04 \\ -1 & s & 0 & 0 \\ 0 & 32 & s & -20 \\ -1 & 0 & 0 & s + 0.8 \end{bmatrix} \tag{61}$$

$$[Is - F]^{-1} = \frac{\begin{bmatrix} s^2(s + 0.8) & 0 & 0 & -1.04s^2 \\ s(s + 0.8) & s(s^2 + 2s + 2) & 0 & -1.04s \\ -(12s + 25.6) & -32(s^2 + 2s + 2) & s(s^2 + 2s + 2) & 20(s^2 + 1.2s + 1.664) \\ s^2 & 0 & 0 & s^2(s + 1.2) \end{bmatrix}}{s^2(s^2 + 2s + 2)} \tag{62}$$

$$[Is - F]^{-1}G = \frac{\begin{bmatrix} 0 & -3s^2(s + 0.8) \\ 0 & -3s(s + 0.8) \\ 10s(s^2 + 2s + 2) & 36s + 76.8 \\ 0 & -3s^2 \end{bmatrix}}{s^2(s^2 + 2s + 2)} \tag{63}$$

Note that the deletion of the extremely small terms in the F matrix of Eq. (59) has resulted in phugoid poles at the origin. This forces us to consider double roots. The short period roots, for the fictitious plane, are at $s = -1 \pm j$, and this forces us to consider complex open-loop poles.

Suppose now that the designer requires closed-loop poles at

$$\text{phugoid} \qquad s = -0.2 \pm j.2 \tag{64}$$

$$\text{short period} \qquad s = -2 \pm j2 \tag{65}$$

Thus, it is required that

$$\Delta(s) = (s^2 + 4s + 8)(s^2 + 0.4s + 0.08) \tag{66}$$

Is it possible to select a set of Q and R that will yield this closed-loop set of poles? Consider the Q abd R matrices, which have the form

$$Q = \begin{bmatrix} q_{11} & q_{12} & q_{13} & q_{14} \\ q_{12} & q_{22} & q_{23} & q_{24} \\ q_{13} & q_{23} & q_{23} & q_{34} \\ q_{14} & q_{24} & q_{34} & q_{44} \end{bmatrix} \tag{67}$$

$$R = \begin{bmatrix} r_{11} & r_{12} \\ r_{12} & r_{22} \end{bmatrix} \tag{68}$$

Note that 13 independent parameters are available to force the required condition of Eq. (66), and it would seem that there must be an infinity of combinations of the q's and r's which would force the desired closed-loop poles. For the time being, let the exact specification of this set remain unspecified.

Now using Eq. (35),

$$WU_0 + B \neq \infty \qquad \text{when} \qquad D = 0 \tag{69}$$

to obtain half of the necessary equations after letting

$$\left.\begin{aligned} \delta_T &= (a_0s^3 + a_1s^2 + a_2s + a_3)/\Delta \\ \delta_e &= (b_0s^3 + b_1s^2 + b_2s + b_3)/\Delta \end{aligned}\right\} \tag{70}$$

Using Eqs. (62, 63, 66, and 70), the equations for various components of the state vector become

$$\dot{\theta}(s) = \frac{-3(s + 0.8)[b_0s^3 + b_1s^2 + b_2s + b_3] + \Delta[(s + 0.8)\dot{\theta}(0) - 1.04\alpha(0)]}{(s^2 + 2s + 2)\Delta} \tag{71}$$

$$\theta(s) = \frac{-3(s + 0.8)[b_0s^3 + b_1s^2 + b_2s + b_3] + \Delta[(s + 0.8)\dot{\theta}(0) + (s^2 + 2s + 2)\theta(\theta) - 1.04\alpha(0)]}{s(s^2 + 2s + 2)\Delta} \tag{72}$$

$$V(s) = \frac{10s(s^2 + 2s + 2)(a_0s^3 + a_1s^2 + a_2s + a_3) + (36s + 76.8)(b_0s^3 + b_1s^2 + b_2s + b_3)}{s^2(s^2 + 2s + 2)\Delta} + \frac{\Delta[-(12s + 25.6)\dot{\theta}(0) - 32(s^2 + 2s + 2)\theta(0) + s(s^2 + 2s + 2)V(0) + 20(s^2 + 1.2s + 1.664)\alpha(0)]}{s^2(s^2 + 2s + 2)\Delta} \quad (73)$$

$$\alpha(s) = \frac{-3(b_0s^3 + b_1s^2 + b_2s + b_3) + \Delta[\dot{\theta}(0) + (s + 1.2)\alpha(0)]}{(s^2 + 2s + 2)\Delta} \quad (74)$$

In the previous example, it was necessary to consider only one component of the state vector when it was required that the open-loop poles D cancel into the numerator. Now, because of the many terms deleted from the equations of motion, it is seen that not all the poles of D are present in every component. The reader may, however, easily verify the following: 1) requiring that $s^2 + 2s + 2$ be a factor of the numerators of Eqs. (71–74) results in the same equation (with a real and imaginary part); 2) requiring that s be a factor of the numerators of Eqs (72) and (73) gives only one equation;)3 the requirement that Eq. (73) contain s^2 forces the derivative of the numerator or Eq. (73) to contain a free s, and this gives the fourth equation. To generate four equations in the eight unknowns let $s = 0$ in numerator Eq. (73), $s = 0$ derivative of numerator of Eq. (73), and $s = -1 + j$ in numerator of Eq. (74). The result, in matrix form and after equating real-to-real and imaginary-to-imaginary when $s = -1 + j$, gives the vector equation

$$A_1a + A_2b = A_3x(0) \quad (75)$$

where

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 3.84 & 1.8 \\ 2 & 0 & -1 & 1 \\ 2 & -2 & 1 & 0 \end{bmatrix} \quad (76)$$

and

$$A_3 = \begin{bmatrix} 0.64 & 1.6 & 0 & -0.832 \\ 4.8896 & 13.312 & -0.064 & -6.62528 \\ 0.64 & 0 & 0 & 2.47466 \\ -2.34666 & 0 & 0 & 0.170668 \end{bmatrix} \begin{matrix} \leftarrow s = 0 \\ \leftarrow s^2 = 0 \\ \left.\begin{matrix} \\ \\ \end{matrix}\right\} \leftarrow s = -1 + j \end{matrix} \quad (77)$$

Also

$$a = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \qquad b = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} \qquad x(0) = \begin{bmatrix} \dot{\theta}(0) \\ \theta(0) \\ v(0) \\ \alpha(0) \end{bmatrix} \quad (78)$$

Turning now to Eq. (36),

$$[R + W_*QW]U_0 \neq \infty \qquad \text{when} \qquad \Delta = 0$$

the first problem becomes one of fixing the values of Q and R which force

$$\Delta(s) = (s^2 + 4s + 8)(s^2 + 0.4s + 0.08)$$

However, since the $\Delta(s)$ of Eq. (66) is required, there is no real need to find the Q's and R's explicity. Consider any component of the $Z(s)$ vector, which has the form

$$f_{11}(Q,R,s)(a_0s^3 + a_1s^2 + a_2s + a_3) + f_{12}(Q,R,s)(b_0s^3 + b_1s^2 + b_2s + b_3) = 0 \qquad \text{when} \qquad \Delta = 0 \quad (79)$$

In Eq. (79), it is necessary only to know that there are enough free parameters in the Q and R matrices to force any number for f_{11} and f_{12}. Thus, we may as well let f_{11} and f_{12} be the parameters. Rather than do this, divide Eq. (79) by f_{11} and call the new parameter f:

$$(a_0s^3 + a_1s^2 + a_2s + a_3) + f(Q,R,s) \times (b_0s^3 + b_1s^2 + b_2s + b_3) = 0 \qquad \text{when} \qquad \Delta = 0 \quad (80)$$

In Eq. (80), let $s = -2 \pm j2$,

$$a_0(16 + 16j) - 8ja_1 + (-2 + j2)a_2 + a_3 + (f_1 + jf_2)[b_0(16 + 16j) - 8jb_1 + (-2 + j2)b_2 + b_3] = 0 \quad (81)$$

Let $s = -0.2 + j.2$

$$a_0(0.016 + 0.016j) - 0.08ja_1 + (-0.2 + j\cdot 2)a_2 + a_3 + (f_3 + jf_4)[b_0(0.016 + 0.016j) - 0.08jb_1 + (-0.2 + j0.2)b_2 + b_3] = 0 \quad (82)$$

Placing Eqs. (81) and (82) in matrix form gives

$$A_4a + A_5A_4b = 0 \quad (83)$$

where

$$A_4 = \begin{bmatrix} 16 & 0 & -2 & 1 \\ 16 & -8 & 2 & 0 \\ 0.016 & 0 & -0.2 & 1 \\ 0.016 & -0.08 & 2 & 0 \end{bmatrix} \qquad A_5 = \begin{bmatrix} f_1 & -f_2 & 0 & 0 \\ f_2 & f_1 & 0 & 0 \\ 0 & 0 & f_3 & -f_4 \\ 0 & 0 & f_4 & -f_3 \end{bmatrix} \quad (84)$$

To summarize, we may pick any set of real numbers to fill in the A_5 matrix of Eqs. (83) and (84). Then solving Eqs. (83) and (75) will force a set of feedback gains which is guaranteed to give the set of required closed-loop poles. For example, in Eq. (84), let $f_1 = f_2 = f_3 = f_4 = 0$. Therefore, $s = 0$ and there is no feedback from the throttle. Solving Eq. (75) gives δ_T as the product of a row vector and the initial condition vector:

$$\delta_e = \frac{\begin{bmatrix} 0.8s^3 + 2.56s^2 & 1.34166s^3 & -0.00833s^3 & 0.57833s^3 \\ & +2.95s^2 & & +(-0.3467s^2) \\ +1.17333s & & +(0.016666s^2) & \\ & +3.2166s & & +(-1.59533s) \\ +0.21333 & +0.53333 & +(-0.16666s) & +(-0.277333) \end{bmatrix}}{(s^2 + 4s + 8)(s^2 + 0.4s + 0.08)} \begin{bmatrix} \dot{\theta}(0) \\ \theta(0) \\ V(0) \\ \alpha(0) \end{bmatrix} \tag{85}$$

By inspection, the feedback gains are

$$K = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -0.8 & -1.34166 & 0.008333 & -0.57833 \end{bmatrix} \tag{86}$$

The equation for the closed-loop system

$$[Is - F + GK]x(s) = x(0)$$

becomes

$$\begin{bmatrix} s + 3.6 & 4.025 & -0.025 & 2.775 \\ -1 & s & 0 & 0 \\ 0 & 32 & s & -20 \\ -1 & 0 & 0 & s + 0.8 \end{bmatrix} \begin{bmatrix} \dot{\theta}(s) \\ \theta(s) \\ V(s) \\ \alpha(s) \end{bmatrix} = \begin{bmatrix} \dot{\theta}(0) \\ \theta(0) \\ V(0) \\ \alpha(0) \end{bmatrix}$$

As a check, it may be readily verified that the determinant of $[I_s - F + GK]$ is

$$(s^2 + 4s + 8)(s^2 + 0.4s + 0.08)$$

In a like manner, one may set $f_2 = f_4 = 0$ and let $f_1, f_3 \to \infty$. This forces $b \equiv 0$, and the case for which there is no feedback from the elevator arises. From Eqs. (75) and (76), it is apparent that no amount of feedback from the throttle alone can force the desired set of closed-loop poles, since A_1^{-1} does not exist. If the small terms in Eq. (59) had been retained, one would find that the system could be controlled with the throttle alone, although the feedback gains would be extremely high. Some indication of these high gains can be seen by picking another set of f's in the A_4 matrix. For example, let

$$f_1 = 10 \qquad f_2 = 100 \qquad f_3 = 1 \qquad f_4 = 5$$

The result is, after solving Eqs. (75 and 83) on the digital computer:

$$\delta_T = \frac{\begin{bmatrix} -43.13199s^3 & 5.6757944s^3 & -0.06331544s^3 & 37.844347s^3 \\ +9.014686s^2 & +89.024725s^2 & +(-0.5228976s^2) & +135.00209s^2 \\ +5.300476s & +28.281683s & +(-0.22238s) & +44.972384s \\ +1.6208835 & +6.409362 & +(-0.047120266) & +5.6800949 \end{bmatrix}}{(s^2 + 4s + 8)(s^2 + 0.4s + 0.08)} \begin{bmatrix} \dot{\theta}(0) \\ \theta(0) \\ V(0) \\ \alpha(0) \end{bmatrix} \tag{87}$$

$$\delta_e = \frac{\begin{bmatrix} 0.5889475s^3 & 0.5071143s^3 & -0.002197882s^3 & -0.16126574s^3 \\ +2.137892s^2 & +1.2808952s^2 & +(-0.00439576s^2) & +(-1.783862s^2) \\ +0.7512283s & +1.5475619s & +(-0.004395764s) & +(-3.0745247s) \\ +0.21333 & +0.5333 & & +(-0.277333) \end{bmatrix}}{(s^2 + 4s + 8)(s^2 + 0.4s + 0.08)} \begin{bmatrix} \dot{\theta}(0) \\ \theta(0) \\ V(0) \\ \alpha(0) \end{bmatrix} \tag{88}$$

By inspection, the feedback gains are

$$K = \begin{bmatrix} 43.13199 & -5.675944 & 0.06331544 & -37.844347 \\ -0.5889475 & -0.5071143 & 0.002197882 & 0.16126574 \end{bmatrix} \tag{89}$$

The closed-loop system,

$$[Is - F + GK]x(s) = x(0)$$

becomes

$$\begin{bmatrix} s + 2.9667 & 1.52133 & -0.0065934 & 0.556205 \\ -1 & s & 0 & 0 \\ 431.32 & -24.758 & s + 0.63315 & -398.44 \\ -1 & 0 & 0 & s + 0.8 \end{bmatrix} \begin{bmatrix} \dot{\theta}(s) \\ \theta(s) \\ V(s) \\ \alpha(s) \end{bmatrix} = \begin{bmatrix} \dot{\theta}(0) \\ \theta(0) \\ V(0) \\ \alpha(0) \end{bmatrix} \tag{90}$$

Again, one may verify that $\det[Is - F + GK]$ is equal to the Δ of Eq. (66).

The result demonstrated in this example is a general one for the two-controller problem. Namely, if one is truly interested in attaining a prescribed set of closed-loop poles, he may proceed directly to setting up equations like (75) and (83), ignoring the question of which precise values of Q and R achieve this. He may instead choose parameters of the A_5 matrix and see directly what effect these choices have on the zeros of the optimal control. This, of course, gives the effect of the choice on the feedback gains at the same time.

References

[1] Rynaski, E. G. and Whitbeck, R. F., "Theory and Application of Linear Optimal Control," Rept. IH-1943-F-1, Oct. 1965, Cornell Aeronautical Lab.

[2] Chang, S. S. L., *Synthesis of Optimum Control Systems,* McGraw-Hill, New York, 1961.

[3] Rynaski, E. G., Whitbeck, R. F., and Dolbin, B. H., "Sensitivity Considerations in the Optimal Control of a Flexible Launch Vehicle," Rept. BE-2311-F-1, June 1967, Cornell Aeronautical Lab.